Problem of the Month: *Lyle's Triangles*

The Problems of the Month (POM) are used in a variety of ways to promote problem-solving and to foster the first standard of mathematical practice from the Common Core State Standards: "Make sense of problems and persevere in solving them." The POM may be used by a teacher to promote problem-solving and to address the differentiated needs of her students. A department or grade level may engage their students in a POM to showcase problem-solving as a key aspect of doing mathematics. It can also be used schoolwide to promote a problem-solving theme at a school. The goal is for all students to have the experience of attacking and solving non-routine problems and developing their mathematical reasoning skills. Although obtaining and justifying solutions to the problems is the objective, the process of learning to problem-solve is even more important.

The Problem of the Month is structured to provide reasonable tasks for all students in a school. The structure of a POM is a shallow floor and a high ceiling, so that all students can productively engage, struggle, and persevere. The Primary Version Level A is designed to be accessible to all students and especially the key challenge for grades K – 1. Level A will be challenging for most second and third graders. Level B may be the limit of where fourth and fifth grade students have success and understanding. Level C may stretch sixth and seventh grade students. Level D may challenge most eighth and ninth grade students, and Level E should be challenging for most high school students. These grade- level expectations are just estimates and should not be used as an absolute minimum expectation or maximum limitation for students. Problem-solving is a learned skill, and students may need many experiences to develop their reasoning skills, approaches, strategies, and the perseverance to be successful. The Problem of the Month builds on sequential levels of understanding. All students should experience Level A and then move through the tasks in order to go as deeply as they can into the problem. There will be those students who will not have access into even Level A. Educators should feel free to modify the task to allow access at some level.

Overview:
In the Problem of the Month *Lyle's Triangles,* students use properties of two-dimensional geometry to solve problems involving spatial visualization and counting. The mathematical topics that underlie this POM are the attributes of polygons, symmetry, spatial visualization, transformations, designing patterns and shapes, using counting principles, similarity, ratios of geometric figures, and creating a general functional relationship to model the composition of geometric figures.

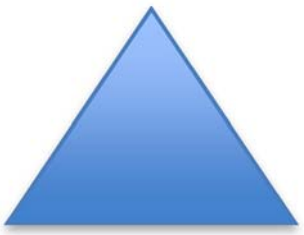
The problem asks the student to use spatial reasoning and attributes to make sense of geometric figure. In Level A of the POM, students are asked to compose an eight-pointed star using 16 congruent right triangles. Students must apply spatial visualization skills to design the figure. The students will use a guess-and-check method and rigid transformation movement to create their arrangements. In level B, students are asked to use four congruent right triangles to create different arrangements. The students need to visualize and categorize the arrangements in an effort to find all possible arrangements. Students are asked to justify how they know they have them all. In level C, students are given a six-pointed star. The students are asked to determine the ratio of the area and length of equilateral triangles that compose the star. In level D, the students explore how to create convex and concave quadrilaterals. They need to determine where to locate the vertices to form both convex and concave quadrilaterals. In level E, students generalize the relationship between the number of unit equilateral triangles that comprise a hexagon and the size of the hexagon, given any size. The student develops a functional relationship between the number of triangles and the size of the hexagon.

Mathematical Concepts:

Spatial visualization plays an important part in real-world experiences. From the most complex structures created by designers, architects, and construction workers to arranging the furniture in a room, spatial awareness and visualization is essential. In this POM, students explore various aspects of spatial visualization. This involves examining symmetrical patterns as well as designing systematic ways to classify or categorize geometric arrangement of figures. Students will use their spatial sense and develop understanding of attributes in plane geometry. They will also design new geometric figures using basic shapes. In addition to the geometric aspects of this POM, the students are seeking to find how many and how much, which requires skills in counting. The students will need to determine the perimeter, area, and ratio of the sizes of figures and determine how to compose and decompose shapes within a figure. The combination of spatial visualization, knowledge of geometric attributes, and counting principles will be called upon to solve the problems in this POM.

Problem of the Month

Lyle's Triangles



Level A

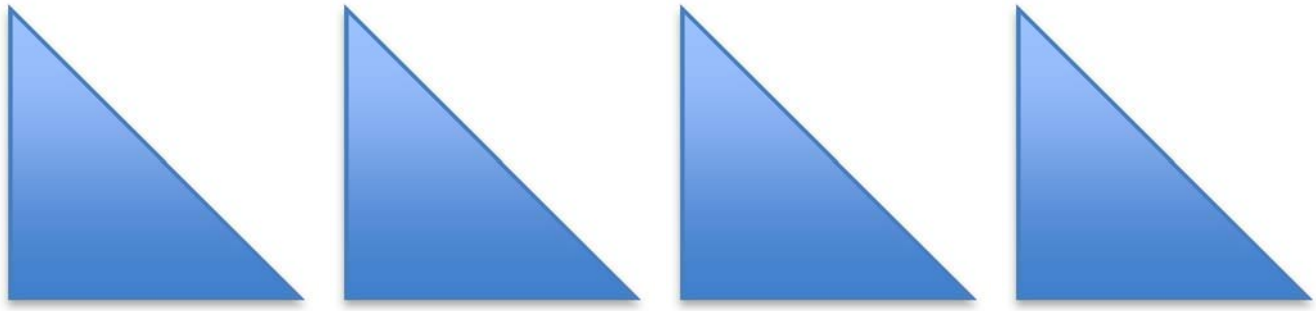
Lyle loves to solve problems. He cut out 16 right triangles all the same size.

Using all 16 triangles, make an eight-pointed star. Make sure all sides touch each other and have no two pieces overlapping.

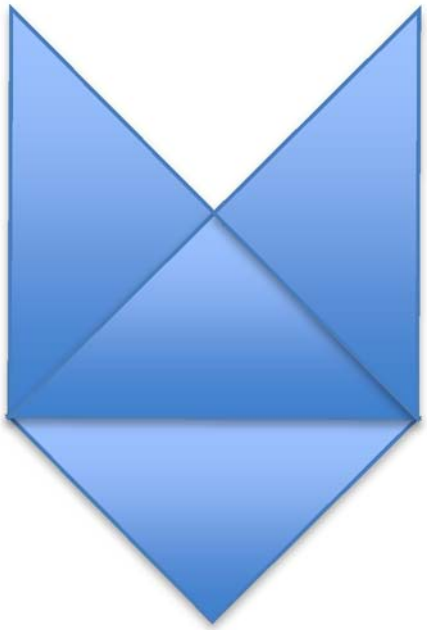
Make a drawing of your star and explain how you created it.

Level B

Lyle cut out four isosceles right triangles.



He arranged the four triangles and made the following shape. Lyle named that shape "Cat." He used all four triangles and made sure each piece touched another piece along a side.



Lyle's Rules for Making a Shape:

1. All four triangles must be used.
2. Each piece must touch at least one other piece.
3. Pieces must touch along a side of the same length.
4. No two pieces may overlap.

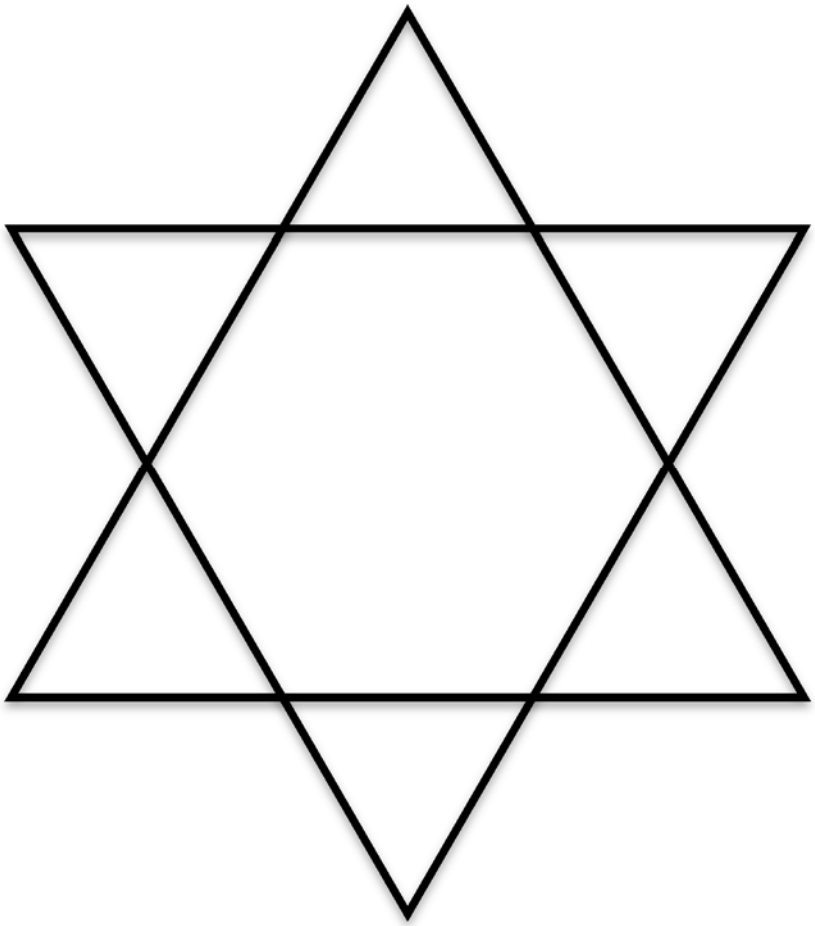
Investigate all the other shapes you can make using the same rules.

How many did you find? What are their names?

How do you know you found all of them?

Level C

Lyle drew a six-pointed star. The six-pointed star consists of two interlocking equivalent triangles. What is the ratio of the area of the entire star to the area of one of the small equilateral triangles?



Explain how you figured it out.

If the length of the small triangle is one unit long, then how many equilateral triangles of any length can be found in the six-pointed star?
Explain your reasoning.

Level D

Three points not in a straight line make up a triangle. Lyle asks, "Where are the locations I should put a fourth point to make sure I have a convex quadrilateral?" Explain.

He also asks, "And where are the locations I should put the fourth point to make a concave quadrilateral?" Explain.

Lyle finally asks, "Where are the locations I should put the fourth point not to make a quadrilateral at all?" Explain.

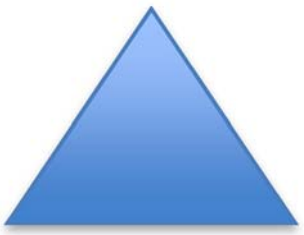
Level E

Lyle says, “Try my super problem!”

A regular hexagon can be divided into equivalent triangles by connecting the vertices. Given any regular hexagon of side length n, how many equilateral triangles of side length 1 can be formed?”

Problem of the Month

Lyle's Triangles



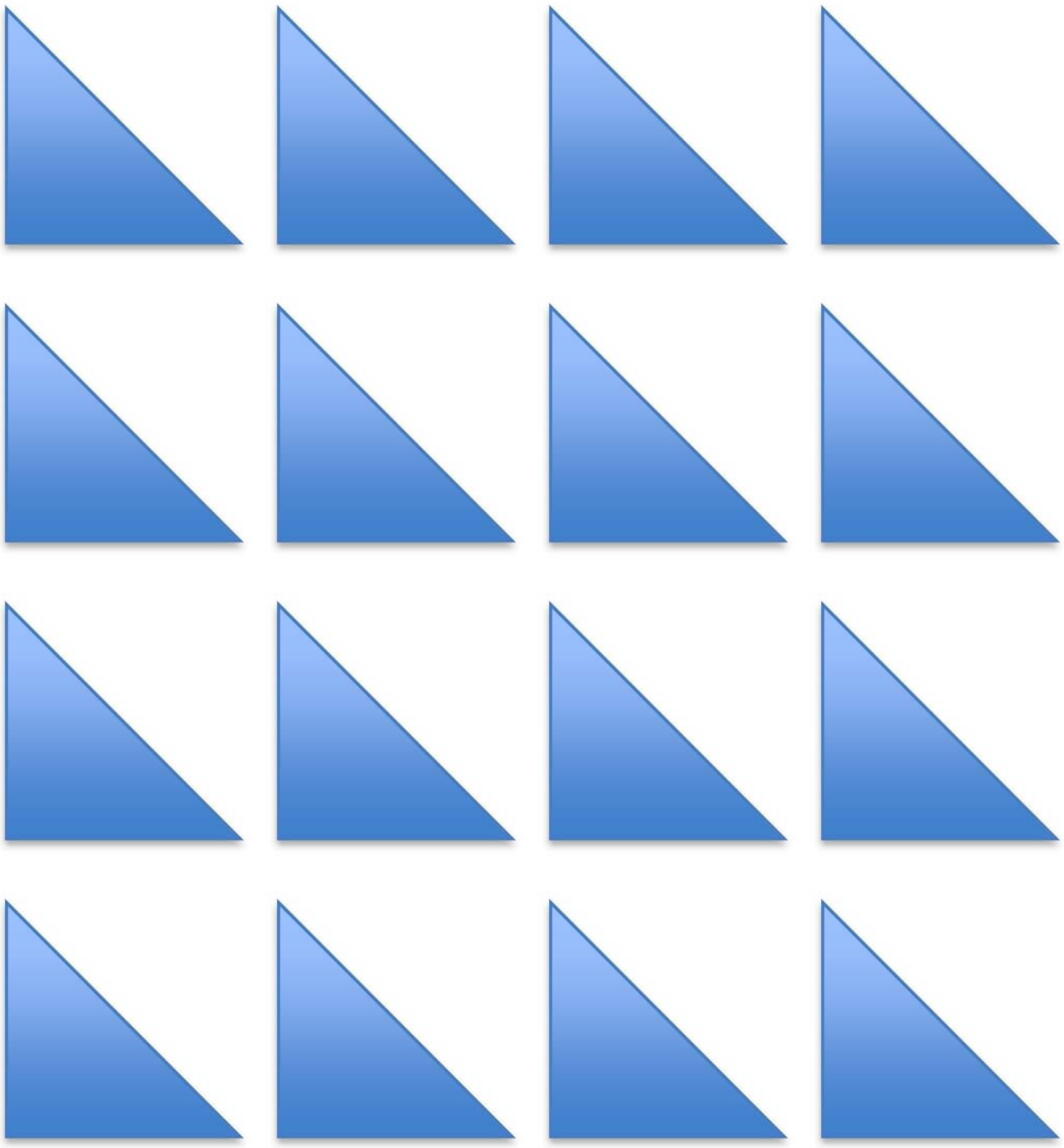
Primary Version Level A

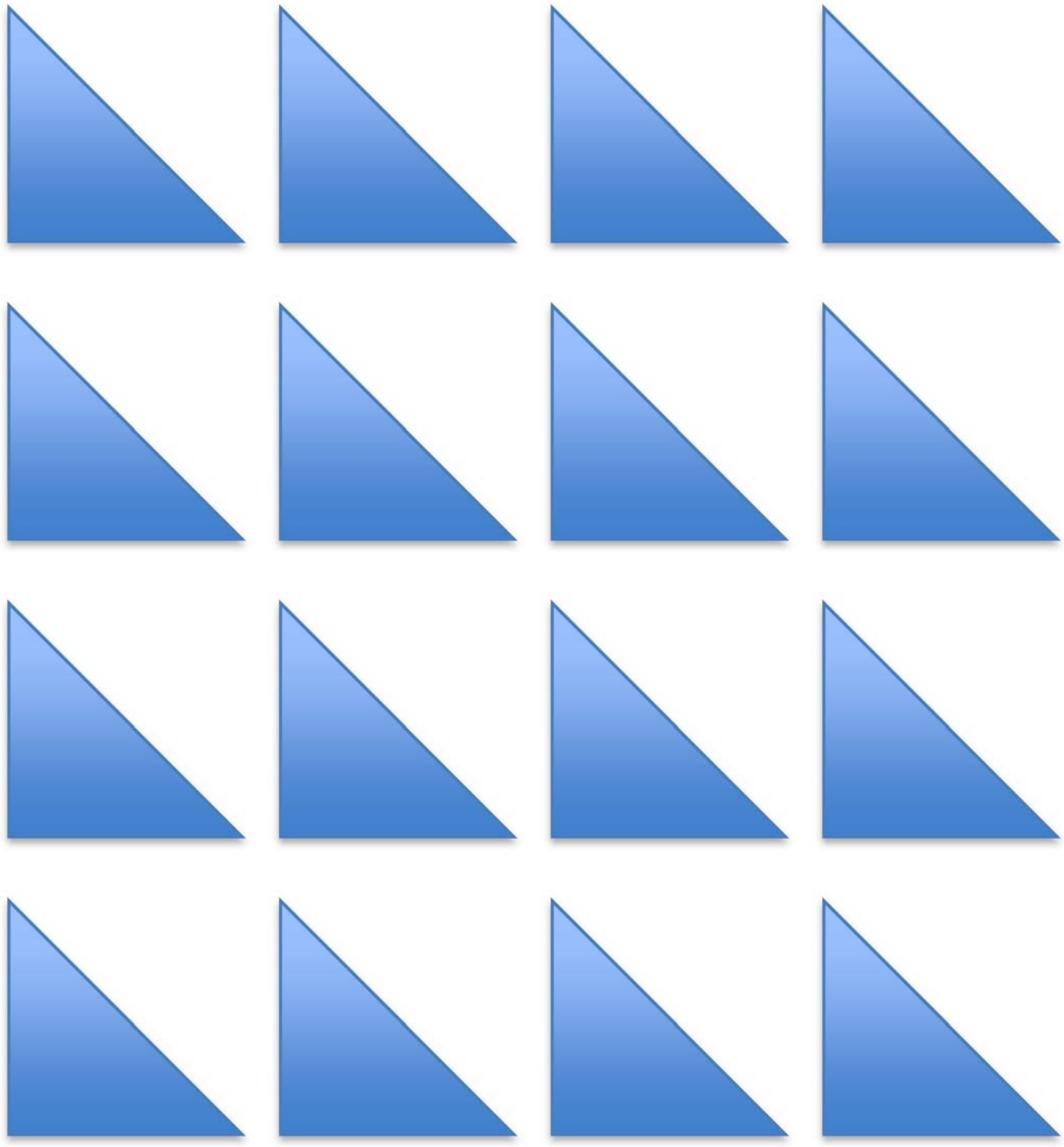
Materials: For the teacher: Ten right triangles, a pair of scissors. For each student: A baggie of ten right triangles, paper, and pencil.

Discussion on the rug: (Teacher shows a triangle.) "What do we call this object?" (Students volunteer names. If students don't know, then the teacher states, "We call it a triangle.") "What do you notice about its shape?" (Students respond with ideas.) Teacher asks, "How can I use some of these to make a square?" (Teacher calls on students to demonstrate and explain.) Teacher asks, "How many pieces did we use? How do we know it is a square? Can we make a square a different way?" (Class explores these ideas.)

In small groups: (Each student has the materials)
The teacher says to the class, "Can we use these pieces to make a rectangle that is not a square? How many pieces are needed to make your rectangle? Draw a picture of your rectangle and show the number of pieces." (Students explore making a rectangle.) Teacher says to the class, "Can we use these pieces to make a bigger triangle? Try making a bigger triangle. How many pieces are needed to make your triangle? Draw a picture of your triangle and show the number of pieces." (Students explore making a triangle.)

(At the end of the investigation, have students either discuss or dictate a response to this summary question.) "How many pieces did you need? Tell me how you figured it out."





Problem of the Month

Lyle's Triangles

Task Description – Level A

This task asks students to use spatial reasoning and geometric attributes to make sense of a figure. The students are asked to compose an eight-pointed star using 16 congruent right triangles. Students must apply spatial visualization skills to design the figure. The students will use a guess-and-check method and rigid transformation movement to create their arrangements.

Common Core State Standards Math - Content Standards

Geometry

Reason with shapes and their attributes.

1.G.1. Distinguish between defining attributes versus non-defining attributes; build and draw shapes to possess defining attributes.

1.G.2. Compose two-dimensional shapes (rectangles, squares, trapezoids, triangles, half-circles, and quarter-circles) or three-dimensional shapes (cubes, right rectangular prisms, right circular cones, and right circular cylinders) to create a composite shape, and compose new shapes from the composite shape.

2.G.1. Recognize and draw shapes having specified attributes, such as a given number of angles or a given number of equal faces.

Common Core State Standards Math – Standards of Mathematical Practice

MP.1 Make sense of problems and persevere in solving them.

Mathematically proficient students start by explaining to themselves the meaning of a problem and looking for entry points to its solution. They analyze givens, constraints, relationships, and goals. They make conjectures about the form and meaning of the solution and plan a solution pathway rather than simply jumping into a solution attempt. They consider analogous problems, and try special cases and simpler forms of the original problem in order to gain insight into its solution. They monitor and evaluate their progress and change course if necessary. Older students might, depending on the context of the problem, transform algebraic expressions or change the viewing window on their graphing calculator to get the information they need. Mathematically proficient students can explain correspondences between equations, verbal descriptions, tables, and graphs or draw diagrams of important features and relationships, graph data, and search for regularity or trends. Younger students might rely on using concrete objects or pictures to help conceptualize and solve a problem. Mathematically proficient students check their answers to problems using a different method, and they continually ask themselves, "Does this make sense?" They can understand the approaches of others to solving complex problems and identify correspondences between different approaches.

MP.7 Look for and make use of structure.

Mathematically proficient students look closely to discern a pattern or structure. Young students, for example, might notice that three and seven more is the same amount as seven and three more, or they may sort a collection of shapes according to how many sides the shapes have. Later, students will see 7 x 8 equals the well-remembered 7 x 5 + 7 x 3, in preparation for learning about the distributive property. In the expression $\mathbf{x}^2 + 9\mathbf{x} + 14$, older students can see the 14 as 2 x 7 and the 9 as 2 + 7. They recognize the significance of an existing line in a geometric figure and can use the strategy of drawing an auxiliary line for solving problems. They also can step back for an overview and shift perspective. They can see complicated things, such as some algebraic expressions, as single objects or as being composed of several objects. For example, they can see $5 - 3(\mathbf{x} - \mathbf{y})^2$ as 5 minus a positive number times a square and use that to realize that its value cannot be more than 5 for any real numbers **x** and **y**.

Problem of the Month

Lyle's Triangles

Task Description – Level B

This task challenges a student to visualize and design different figures composed of right triangles. The students are asked to use four congruent right triangles to create different arrangements. The students need to visualize and categorize the arrangements in an effort to find all possible arrangements. Students are asked to justify how they know they have them all.

Common Core State Standards Math - Content Standards

Geometry

Reason with shapes and their attributes.

1.G.1. Distinguish between defining attributes versus non-defining attributes; build and draw shapes to possess defining attributes.

1.G.2. Compose two-dimensional shapes or three-dimensional shapes to create a composite shape, and compose new shapes from the composite shape.

2.G.1.Recognize and draw shapes having specified attributes, such as a given number of angles or a given number of equal faces.

3.G.2. Partition shapes into parts with equal areas. Express the area of each part as a unit fraction of the whole.

Common Core State Standards Math – Standards of Mathematical Practice

MP.3 Construct viable arguments and critique the reasoning of others.

Mathematically proficient students understand and use stated assumptions, definitions, and previously established results in constructing arguments. They make conjectures and build a logical progression of statements to explore the truth of their conjectures. They are able to analyze situations by breaking them into cases, and can recognize and use counterexamples. They justify their conclusions, communicate them to others, and respond to the arguments of others. They reason inductively about data, making plausible arguments that take into account the context from which the data arose. Mathematically proficient students are also able to compare the effectiveness of two plausible arguments, distinguish correct logic or reasoning from that which is flawed, and—if there is a flaw in an argument—explain what it is. Elementary students can construct arguments using concrete referents such as objects, drawings, diagrams, and actions. Such arguments can make sense and be correct, even though they are not generalized or made formal until later grades. Later, students learn to determine domains to which an argument applies. Students at all grades can listen or read the arguments of others, decide whether they make sense, and ask useful questions to clarify or improve the arguments.

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Problem of the Month

Lyle's Triangles

Task Description – Level C

This task challenges a student to determine the ratios of areas in a six-pointed star. The students are given a six-pointed star. The students are asked to determine the ratio of the area and length of equilateral triangles that compose the star.

Common Core State Standards Math - Content Standards

Geometry

Reason with shapes and their attributes.

3.G.2. Partition shapes into parts with equal areas. Express the area of each part as a unit fraction of the whole.

Draw and identify lines and angles, and classify shapes by properties of their lines and angles.

4.G.1. Draw points, lines, line segments, rays, angles (right, acute, obtuse), and perpendicular and parallel lines. Identify these in two-dimensional figures.

Ratios and Proportional Relationships

Understand ratio concepts and use ratio reasoning to solve problems.

6.RP.1. Understand the concept of a ratio and use ratio language to describe a ratio relationship between two quantities.

6.RP.3. Use ratio and rate reasoning to solve real-world and mathematical problems.

Common Core State Standards Math – Standards of Mathematical Practice

MP.2 Reason abstractly and quantitatively.

Mathematically proficient students make sense of quantities and their relationships in problem situations. They bring two complementary abilities to bear on problems involving quantitative relationships: the ability to decontextualize—to abstract a given situation and represent it symbolically and manipulate the representing symbols as if they have a life of their own, without necessarily attending to their referents—and the ability to contextualize, to pause as needed during the manipulation process in order to probe into the referents for the symbols involved. Quantitative reasoning entails habits of creating a coherent representation of the problem at hand; considering the units involved; attending to the meaning of quantities, not just how to compute them; and knowing and flexibly using different properties of operations and objects.

MP.4 Model with mathematics.

Mathematically proficient students can apply the mathematics they know to solve problems arising in everyday life, society, and the workplace. In early grades, this might be as simple as writing an addition equation to describe a situation. In middle grades, a student might apply proportional reasoning to plan a school event or analyze a problem in the community. By high school, a student might use geometry to solve a design problem or use a function to describe how one quantity of interest depends on another. Mathematically proficient students who can apply what they know are comfortable making assumptions and approximations to simplify a complicated situation, realizing that these may need revision later. They are able to identify important quantities in a practical situation and map their relationships using such tools as diagrams, two-way tables, graphs, flowcharts and formulas. They can analyze those relationships mathematically to draw conclusions. They routinely interpret their mathematical results in the context of the situation and reflect on whether the results make sense, possibly improving the model if it has not served its purpose.

Problem of the Month:

Lyle's Triangles

Task Description – Level D

This task challenges a student to use visualization and reasoning to construct and draw quadrilaterals with different attributes. The students explore how to create convex and concave quadrilaterals. They need to determine where to locate the vertices to form both convex and concave quadrilaterals.

Common Core State Standards Math - Content Standards

Geometry

Draw and identify lines and angles, and classify shapes by properties of their lines and angles.
4.G.1.Draw points, lines, line segments, rays, angles (right, acute, obtuse), and perpendicular and parallel lines. Identify these in two-dimensional figures.

Draw, construct, and describe geometrical figures and describe the relationships between them.
7.G.2. Draw (freehand, with ruler and protractor, and with technology) geometric shapes with given conditions.

Solve real-life and mathematical problems involving angle measure, area, surface area, and volume.
7.G.6. Solve real-world and mathematical problems involving area, volume and surface area of two- and three-dimensional objects composed of triangles, quadrilaterals, polygons, cubes, and right prisms.

Common Core State Standards Math – Standards of Mathematical Practice

MP.1 Make sense of problems and persevere in solving them.
Mathematically proficient students start by explaining to themselves the meaning of a problem and looking for entry points to its solution. They analyze givens, constraints, relationships, and goals. They make conjectures about the form and meaning of the solution and plan a solution pathway rather than simply jumping into a solution attempt. They consider analogous problems, and try special cases and simpler forms of the original problem in order to gain insight into its solution. They monitor and evaluate their progress and change course if necessary. Older students might, depending on the context of the problem, transform algebraic expressions or change the viewing window on their graphing calculator to get the information they need. Mathematically proficient students can explain correspondences between equations, verbal descriptions, tables, and graphs or draw diagrams of important features and relationships, graph data, and search for regularity or trends. Younger students might rely on using concrete objects or pictures to help conceptualize and solve a problem. Mathematically proficient students check their answers to problems using a different method, and they continually ask themselves, "Does this make sense?" They can understand the approaches of others to solving complex problems and identify correspondences between different approaches.

MP.7 Look for and make use of structure.
Mathematically proficient students look closely to discern a pattern or structure. Young students, for example, might notice that three and seven more is the same amount as seven and three more, or they may sort a collection of shapes according to how many sides the shapes have. Later, students will see 7 x 8 equals the well-remembered 7 x 5 + 7 x 3, in preparation for learning about the distributive property. In the expression $\mathbf{x}^2 + 9\mathbf{x} + 14$, older students can see the 14 as 2 x 7 and the 9 as 2 + 7. They recognize the significance of an existing line in a geometric figure and can use the strategy of drawing an auxiliary line for solving problems. They also can step back for an overview and shift perspective. They can see complicated things, such as some algebraic expressions, as single objects or as being composed of several objects. For example, they can see $5 - 3(\mathbf{x} - \mathbf{y})^2$ as 5 minus a positive number times a square and use that to realize that its value cannot be more than 5 for any real numbers **x** and **y**.

Problem of the Month

Lyle's Triangles

Task Description – Level E

This task challenges a student to analyze a pattern involving geometric shapes and determine a function. The student generalizes the relationship between the number of unit equilateral triangles that comprise a hexagon and the size of the hexagon, given any size. The student develops a functional relationship between the number of triangles and the size of the hexagon.

Common Core State Standards Math - Content Standards

High School – Functions - Building Functions

Build a function that models a relationship between two quantities

F-BF.1. Write a function that describes a relationship between two quantities.

a. Determine an explicit expression, a recursive process, or steps for calculation from a context.

High School – Geometry - Modeling with Geometry

Apply geometric concepts in modeling situations

G-MG.1. Use geometric shapes, their measures, and their properties to describe objects

Common Core State Standards Math – Standards of Mathematical Practice

MP.2 Reason abstractly and quantitatively.

Mathematically proficient students make sense of quantities and their relationships in problem situations. They bring two complementary abilities to bear on problems involving quantitative relationships: the ability to decontextualize—to abstract a given situation and represent it symbolically and manipulate the representing symbols as if they have a life of their own, without necessarily attending to their referents—and the ability to contextualize, to pause as needed during the manipulation process in order to probe into the referents for the symbols involved. Quantitative reasoning entails habits of creating a coherent representation of the problem at hand; considering the units involved; attending to the meaning of quantities, not just how to compute them; and knowing and flexibly using different properties of operations and objects.a

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Problem of the Month

Lyle's Triangles

Task Description – Primary Level

This task challenges students to use right triangles to make other shapes. The students are asked to use a number of right triangles to make squares, rectangles that are not squares, and larger triangles.

Common Core State Standards Math - Content Standards

Geometry

Identify and describe shapes

K.G.2. Correctly name shapes regardless of their orientations or overall size.

Analyze, compare, create, and compose shapes.

K.G.4. Analyze and compare two- and three-dimensional shapes, in different sizes and orientations, using informal language to describe their similarities, differences, parts.

K.G.5. Model shapes in the world by building shapes from components and drawing shapes.

K.G.6. Compose simple shapes to form larger shapes.

Common Core State Standards Math – Standards of Mathematical Practice

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